Quadratic Eq^n

An eq^n of the form,

$$\boxed{ax^2 + bx + c = 0}$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$.

- If its roots are α, β then

$$\boxed{\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}} \quad \boxed{\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

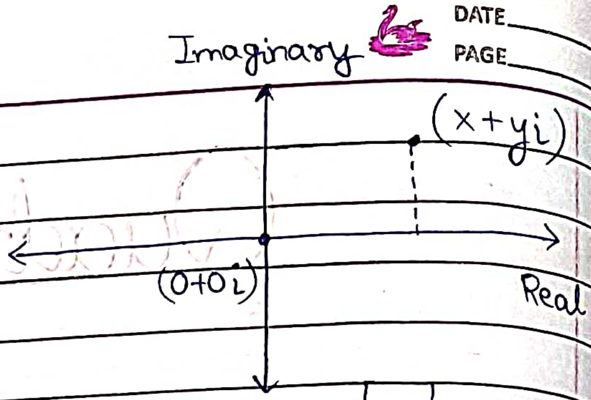
- Nature of Roots —

Let $\sqrt{D} = \sqrt{b^2 - 4ac}$ be discriminant

$\Rightarrow$ $\boxed{D = b^2 - 4ac}$

- $D > 0 \Rightarrow$ Real & Distinct roots
- $D = 0 \Rightarrow$ Real & Equal roots
- $D < 0 \Rightarrow$ Non-real roots

Non-Real Roots :



- Argand plane

- Complex No. — $z = x + yi$; $i = \sqrt{-1}$; $x, y \in \mathbb{R}$

$\Rightarrow$ $\boxed{x = Re(z)}$, $\boxed{y = Im(z)}$

☆ (0+0i) is PURELY REAL as well as PURELY IMAGINARY

- Modulus — $|z|$ = Dist. from $(0+0i)$

$\Rightarrow$ $\boxed{|z| = \sqrt{x^2+y^2}}$

- Conjugate — It is image of complex no. in real axis

$\Rightarrow$ $\boxed{\bar{z} = x - yi}$

Then

— If $p+qi$ ($p, q \in \mathbb{R}$) is a root of eq^n, ~~and~~ then $p-qi$ is also a root; if coeff. of eq^n real!

Then

- If $(p+\sqrt{q})$ is an irrational root of eqⁿ, then and $(p-\sqrt{q})$ is also a root of the eqⁿ, if all coeff. of eqⁿ rational!

- Eqⁿ has rational roots if $D =$ perfect sqᵉ and $a, b, c \in \mathbb{Q}$

- If $a=1$ and $b, c \in \mathbb{Z}$ and roots are rational $\Rightarrow$ roots must be $\mathbb{Z}$

Proof: $x^2+bx+c=0 \Rightarrow$ Roots $= \frac{-b \pm \sqrt{b^2-4c}}{2}$

C1 – $b=$ odd $\Rightarrow (b^2-4c) =$ odd $\Rightarrow$ Root $\in \mathbb{Z}$.

C2 – $b =$ even $\Rightarrow (b^2-4c) =$ even $\Rightarrow$ Root $\in \mathbb{Z}$.

- If eqⁿ has more than 2 nos (real or complex), then it becomes identity.

$$\Rightarrow \boxed{a=b=c=0}$$

- Let α, β be 2 roots of given eqⁿ then

$$\alpha+\beta = \left(-\frac{b}{a}\right) \quad \text{and} \quad \frac{c}{a} = \alpha\beta$$

DATE

PAGE

- Eq^n with roots α, β;

$$\Rightarrow \star \quad \boxed{ax^2 + bx + c = a(x-\alpha)(x-\beta)}$$

- For cubic, $ax^2 + bx^2 + cx + d = 0$ — α, β, γ

$$\Rightarrow \quad \boxed{ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)}$$

$$\Rightarrow \left(\frac{-b}{a}\right) = \sum \alpha, \quad \left(\frac{c}{a}\right) = \sum \alpha\beta, \quad \left(\frac{-d}{a}\right) = \alpha\beta\gamma$$

For $poly^n$, $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ — α_1, $\vdots$, α_n

$$\Rightarrow \quad \boxed{a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = a_n(x-\alpha_1)\dots(x-\alpha_n)}$$

$$\Rightarrow \left(\frac{-a_{n-1}}{a_n}\right) = \sum \alpha_i, \quad \left(\frac{a_{n-2}}{a_n}\right) = \sum \alpha_i \alpha_j, \quad \left(\frac{a_0}{a_n}\right) = (-1)^n \Pi$$

- If $P(a)$ and $P(b)$ are of opp. sign then $P(x)$ has odd no. of roots in $x \in [a, b]$ i.e. $\exists$ at least one root of $P(x)$ inside $x \in [a, b]$

[Intermediate Value Theorem]

- If α is a root of multiplicity 'r' of $f(x)$, then

$$\boxed{f(x) = (x-\alpha)^r g(x)} \quad \text{where} \quad \boxed{g(\alpha) \neq 0}$$

and $\boxed{f(\alpha) = f'(\alpha) = f''(\alpha) = \ldots = f^{(r-1)}(\alpha) = 0}$.

- If coeff. in $p(x)$ have 'm' changes in sign, then $p(x)=0$ have at most 'm' (+ve) real roots

If coeff. in $p(-x)$ have 't' changes in sign, then $p(x)=0$ have at most 't' (-ve) real roots

[Descartes' Rule of Signs]

Q) i) If α, β roots of $c+(x-a)(x-b)=0$, find roots of $(x-\alpha)(x-\beta)=c$.

ii) If roots of $x^2-ax+b=0$ are real and differ by a qty less than c

p.t. $\left(\frac{a^2-c^2}{4}\right) < b < \left(\frac{a^2}{4}\right)$

DATE
PAGE

A) i) $(x-a)(x-b)+c = (x-\alpha)(x-\beta)$

$\Rightarrow (x-\alpha)(x-\beta) = c$ — a, b

Roots : a, b.

ii) Roots : $\alpha, \beta \Rightarrow 0 < |\alpha-\beta| < c$

$\Rightarrow 0 < \alpha^2+\beta^2-2\alpha\beta < c^2$

$\Rightarrow 0 < (\alpha^2+\beta^2+2\alpha\beta) - 4\alpha\beta < c^2$

$\Rightarrow 0 < a^2 - 4b < c^2 \Rightarrow \left(\frac{a^2-c^2}{4}\right) < b < \left(\frac{a^2}{4}\right)$

Q) Let a, b, c be real nos. with $a \neq 0$. If α, β are roots of eq^n $ax^2+bx+c=0$ express roots of $a^3x^2+abcx+c^3=0$ in terms of α, β.

A) $a\left(\frac{ax}{c}\right)^2 + b\left(\frac{ax}{c}\right) + c = 0 \Rightarrow$ Roots : $\left(\frac{c\alpha}{a}\right), \left(\frac{c\beta}{a}\right)$

Q) Consider $x^2+x-n=0$, $n \in \mathbb{Z}$ and $1 \leq n < 100$. Find total no. of diff. values of n s.t. eq^n has int. roots.

A) Roots: $\alpha, -1-\alpha$ $\Rightarrow$ $\alpha^2+\alpha=n$ $\rightarrow$ $\boxed{\text{9 values of } n}$

$\alpha \in \{-10, \dots, -2, 1, \dots, 9\} \Rightarrow$ 18 values of α

Q) Find integral values of 'a' for which roots of $x^2+(a-1)x-(a+2)=0$ are integral.

A) Roots $\in \mathbb{Z} \subseteq \mathbb{R} \Rightarrow (a-1)^2+4(a+2) \geq 0$

$\Rightarrow \quad a^2+2a+5 \geq 0 \Rightarrow$ True $\forall a$.

$\cancel{x(x+(a-1))}$ $x^2+(a-1)x=(a+2)$

$\Rightarrow (x^2-x)+(ax-a)=2 \Rightarrow (x+a)(x-1)=2$

Now, $2 = 2\cdot 1 = 1\cdot 2 = (-1)\cdot(-2) = (-2)\cdot(-1)$

$\Rightarrow$ $x-1 = 1, 2, -1, -2$
$x+a = 2, 1, -2, -1$ $\Rightarrow \boxed{a \in \{0, -2\}}$

Q) Find 'a' s.t. $(a-1)(x^2+x+1)^2=(a+1)(x^4+x^2+1)$ has 2 real distinct roots.

A) $(a-1)(x^2+x+1)^2 = (a+1)\left((x^4+2x^2+1)-x^2\right)$

$= (a+1)(x^2+x+1)(x^2-x+1)$

$\Rightarrow (a-1)(x^2+x+1) = (a+1)(x^2-x+1)$

$\Rightarrow 2x^2 - 2ax + 2 = 0 \Rightarrow \boxed{x^2 - ax + 1 = 0}$

For distinct real roots, $\boxed{a \in (-\infty,-2) \cup (2,\infty)}$

Q) Let $\alpha+\beta i$; $\alpha,\beta \in \mathbb{R}$ be a root of $x^3+qx+r=0$; $q,r \in \mathbb{R}$. Find a real cubic independent of α,β with root 2α.

A) Roots: $\alpha+\beta i$, $\alpha-\beta i$, -2α.

By Vieta's relⁿ, $(-2\alpha)(\alpha^2+\beta^2) = (-r)$

$\Rightarrow 2\alpha(\alpha^2+\beta^2) = r$

$(-2\alpha)(\alpha+\beta i+\alpha-\beta i) + (\alpha^2+\beta^2) = q$

$\Rightarrow \beta^2 - 3\alpha^2 = q$

$\Rightarrow 2\alpha(q+4\alpha^2) = r \Rightarrow \boxed{\text{Cubic: } x^3+qx-r=0}$

Quadratic Expression

Let $\boxed{f(x) = ax^2+bx+c}$

$\Rightarrow \quad f'(x) = 2ax+b \Rightarrow x = \left(\frac{-b}{2a}\right)$ (Critical Pt.)

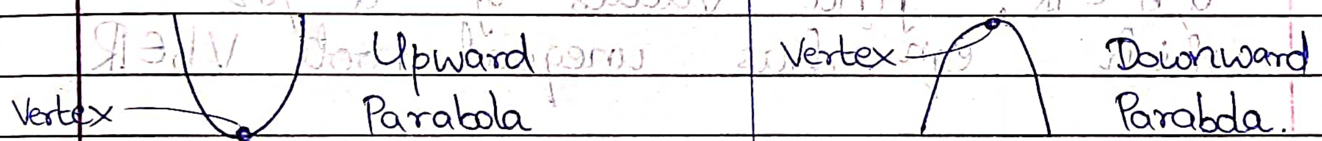
$\Rightarrow \quad f''(x) = 2a$

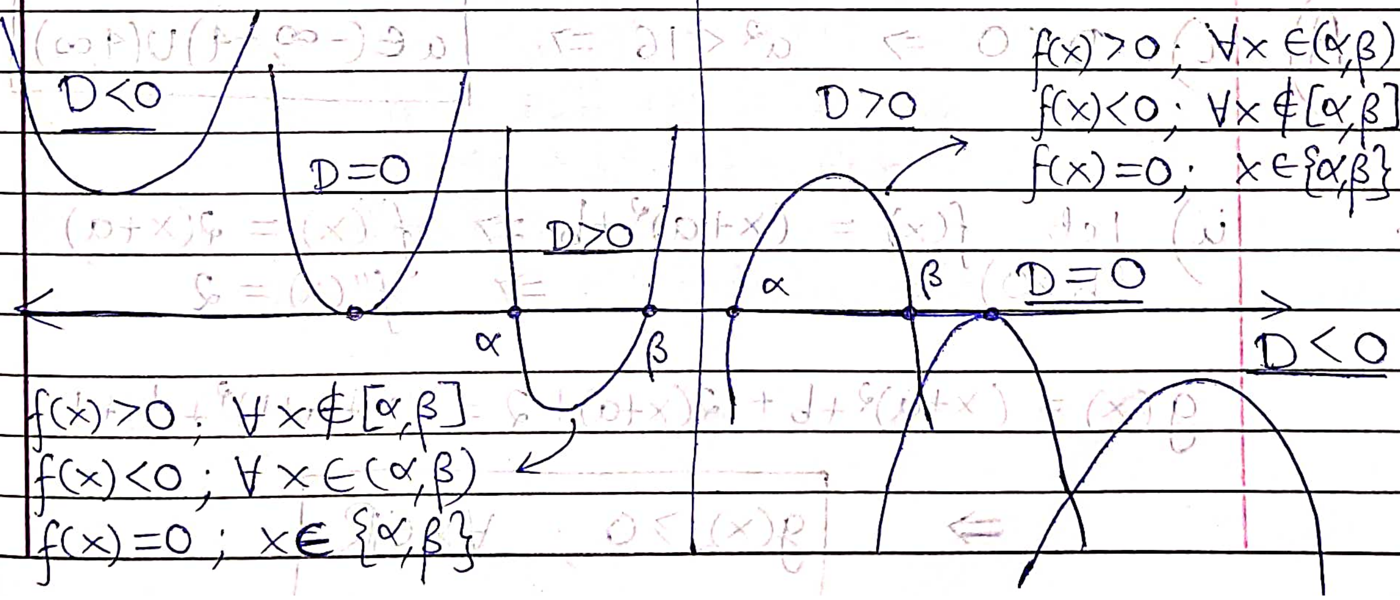
(i) when $a>0 \Rightarrow f''(-b/2a) > 0 \Rightarrow$ Minima

(ii) when $a<0 \Rightarrow f''(-b/2a) < 0 \Rightarrow$ Maxima

At $x = \left(\frac{-b}{2a}\right)$, $f(x) = \left(\frac{-D}{4a}\right)$

$a>0$	$a<0$





$f(x)>0;\ \forall x \in (\alpha,\beta)$
$f(x)<0;\ \forall x \notin [\alpha,\beta]$
$f(x)=0;\ x \in \{\alpha,\beta\}$

$f(x)>0;\ \forall x \notin [\alpha,\beta]$
$f(x)<0;\ \forall x \in (\alpha,\beta)$
$f(x)=0;\ x \in \{\alpha,\beta\}$

Q) (i) If $x^2 - ax + 4 > 0$, $\forall x \in \mathbb{R}$; find 'a'.

(ii) If $f(x)$ is quad. s.t. $f(x) > 0$; $\forall x \in \mathbb{R}$ and $g(x) = f(x) + f'(x) + f''(x)$. Then p.t. $g(x) \geq 0$; $\forall x \in \mathbb{R}$.

★ (iii) If $ax^2 - bx + 5 = 0$ does NOT have 2 real distinct roots, then find min. possible value of $(2a + b)$. (See 184)

(iv) If $c > 0$ and $2ax^2 + 3bx + 5c = 0$ does NOT have any real roots, then p.t. $2a - 3b + 5c > 0$.

(v) Find range of $\left(\dfrac{x^2 + x + 2}{x^2 + x + 1}\right)$

(vi) If $x^2 + (a-b)x + (1-a-b) = 0$, where $a, b \in \mathbb{R}$; find values of 'a' for which eqn has unequal roots $\forall b \in \mathbb{R}$.

A) i) $D < 0 \Rightarrow a^2 < 16 \Rightarrow$ $\boxed{a \in (-\infty, -4) \cup (4, \infty)}$

ii) Let $f(x) = (x+a)^2 + b \Rightarrow f'(x) = 2(x+a)$
$(b > 0)$ $\Rightarrow f''(x) = 2$

$g(x) = (x+a)^2 + b + 2(x+a) + 2 = (x+a+1)^2 + b + 1$

$\Rightarrow \boxed{g(x) \geq 0, \ \forall x \in \mathbb{R}}$

iii) iv) $f(x) = 2ax^2 + 3bx + 5c \Rightarrow f(0) = 5c > 0$

$f(x)$ has <u>NO REAL</u> root & $f(0) > 0 \Rightarrow f(x) > 0, \ \forall x \in \mathbb{R}$

$\Rightarrow \boxed{f(-1) = 2a - 3b + 5c > 0}$

iii) $D \leqslant 0 \Rightarrow b^2 - 20a \leqslant 0 \Rightarrow b^2 + 10b \leqslant 10(2a+b)$

$\Rightarrow 10(2a+b) \geqslant (b+5)^2 - 25 \Rightarrow \boxed{2a+b \geqslant (-2.5)}$

v) $y = \left(\dfrac{x^2+x+2}{x^2+x+1}\right) \Rightarrow x^2(y-1) + x(y-1) + (y-2) = 0$
$\{y \neq 1\}$

for $x \in \mathbb{R} \Rightarrow D \geqslant 0 \Rightarrow (y-1)^2 - 4(y-1)(y-2) \geqslant 0$

$\Rightarrow (y-1)[y-1-4y+8] \geqslant 0$

$\Rightarrow (y-1)(3y-7) \leqslant 0 \Rightarrow \boxed{y \in (1, 7/3]}$

vi) $D > 0 \Rightarrow (a-b)^2 > 4(1-a-b)$

$(\forall b \in \mathbb{R}) \Rightarrow a^2 - 2ab + b^2 > 4 - 4a - 4b$

$\Rightarrow a^2 + 4a + b^2 + 4b - 2ab > 4$

$\cancel{\Rightarrow a^2 + b^2}$

$\Rightarrow b^2 + (4-2a)b + (a^2+4a-4) > 0 \ (\forall b \in \mathbb{R})$

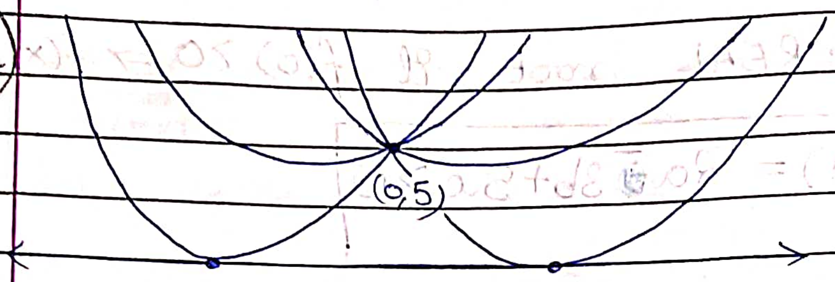
$\Rightarrow D_1 < 0 \Rightarrow (2a-4)^2 < 4(a^2+4a-4)$

$\Rightarrow a^2 - 4a + 4 < a^2 + 4a - 4 \Rightarrow \boxed{a > 1}$

184

★ iii) Draw $f(x) = ax^2 - bx + 5$

(See 182)



$\Rightarrow f(x) \geq 0, \forall x \in \mathbb{R}$

$\Rightarrow ax^2 - bx + 5 \geq 0, \forall x \in \mathbb{R} \xrightarrow{x=(-2)} \boxed{2a + b \geq (-2.5)}$

★ Q) Draw graph of $f(x) = \dfrac{(x+1)(x+3)}{(x-1)(x-2)}$.

A) $f(x) = \left(\dfrac{x^2+4x+3}{x^2-3x+2}\right) = 1 + \left(\dfrac{7x+1}{x^2-3x+2}\right)$

We have asymptotes : $x = 1$, $x = 2$.

We have $f(-1) = f(-3) = 0$.

We have $f(0) = 3/2$.

Now, $\lim\limits_{x \to 1^-} (f(x)) = +\infty$

$\lim\limits_{x \to 1^+} (f(x)) = -\infty$

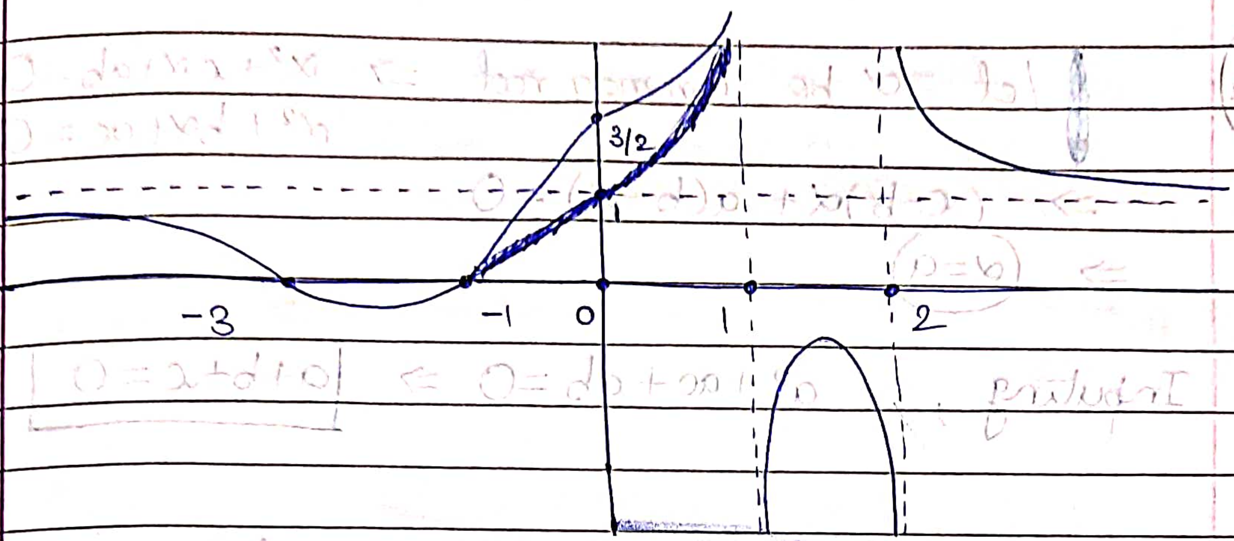
$\lim\limits_{x \to 2^-} (f(x)) = -\infty$

$\lim\limits_{x \to 2^+} (f(x)) = +\infty$

Now, $f(x) < 1$; $\forall x < (-1/7)$

Now, $f(x) > 1$; $\forall x > 2$

Now, $\lim\limits_{x \to \infty} (f(x)) = \lim\limits_{x \to (-\infty)} (f(x)) = 1$



Common Roots

If $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have,

1) Both roots common $\Rightarrow \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

2) Exactly 1 root common $\Rightarrow \left(\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\right)^2 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Proof: Let α be common root.

$\Rightarrow \begin{array}{l} a_1\alpha^2 + b_1\alpha + c_1 = 0 \\ a_2\alpha^2 + b_2\alpha + c_2 = 0 \end{array} \Rightarrow \left(\dfrac{\alpha^2}{b_1c_2 - b_2c_1}\right) = \left(\dfrac{-\alpha}{a_1c_2 - a_2c_1}\right) = \left(\dfrac{1}{a_1b_2 - a_2b_1}\right)$

DATE
PAGE

Exercise I (Module)

Q5) Let α be common root $\Rightarrow$ $\alpha^2+c\alpha+ab=0$
$\alpha^2+b\alpha+ac=0$

$\Rightarrow (c-b)\alpha+a(b-c)=0$

$\Rightarrow \alpha=a$

Inputing, $a^2+ac+ab=0 \Rightarrow \boxed{a+b+c=0}$

Q6) Observe $x=1$ is common root.

Q8) $D = 9-4\cdot5<0 \Rightarrow$ Complex Roots
$\Rightarrow$ Both roots common

$\Rightarrow \left(\frac{a}{1}\right)=\left(\frac{b}{3}\right)=\left(\frac{c}{5}\right) \Rightarrow \boxed{(a+b+c)_{min}=9}$

Q10) Observe $x=1$ is common root.

Let α be common root. $\Rightarrow$ $\alpha^2+a\alpha+b=0$
$\alpha^2+b\alpha+a=0$

$\Rightarrow (a-b)\alpha+(b-a)=0$

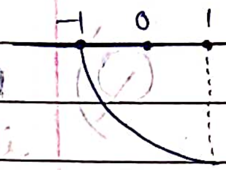
$\Rightarrow \alpha=1$

Q) Find range of $f(x) = \sin^2(x) - 5\sin(x) - 6$.

Q) Find least integral value of a for which $ax^2 + 12x - 3x^2 + 6 + a > 0$, $\forall x \in \mathbb{R}$

A) $f(x) = s_x^2 - 5s_x + \frac{25}{4} - \frac{49}{4} = \left(s_x - \frac{5}{2}\right)^2 - \frac{49}{4}$

$s_x \in [-1, 1] \Rightarrow \left(s_x - 5/2\right) \in \left[-7/2, -3/2\right]$



$\Rightarrow \cancel{f(x) \in [0,}$ $\boxed{f(x) \in [-10, 0]}$

A) $(a-3)x^2 + 12x + (a+6) > 0$, $\forall x \in \mathbb{R}$

$\Rightarrow 144 < 4(a+6)(a-3) \Rightarrow a^2 + 3a - 18 > 36$

$\Rightarrow (a+9)(a-6) > 0 \Rightarrow \cancel{a \in (-9, 6)}$

$\Rightarrow \boxed{a \in (-\infty, -9) \cup (6, \infty)} \rightarrow \boxed{a \in (6, \infty)}$

for (+ve), $(a-3) > 0 \Rightarrow \boxed{a > 3}$

$\Rightarrow \boxed{a_{min} = 7}$

DATE
PAGE

Location of Roots

1) $ax^2 + bx + c = 0 \longrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

2) Draw graph, showing given conditions

3) Apply relevant conditions. $\begin{cases} \text{Discriminant} \\ \text{Fx}^n \text{ value at smpt.} \\ \text{Vertex.} \end{cases}$

Q) Find 'a', $x^2 + 2(a-1)x + (a+5) = 0$
if roots are –

1) real & distinct   2) real & equal

3) not real   4) Opp. in sign

5) equal in magnitude, opp. in sign

A) 1) $D > 0 \Rightarrow 4(a-1)^2 > 4(a+5) \Rightarrow a^2 - 3a - 4 > 0$

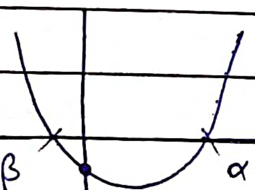
$\Rightarrow \boxed{a \in (-\infty, -1] \cup [4, \infty)}$

2) $D = 0 \Rightarrow 4(a-1)^2 = 4(a+5) \Rightarrow \boxed{a = (-1), 4}$

3) $D < 0 \Rightarrow$ ~~f(0) < 0~~ $4(a-1)^2 < 4(a+5)$

$\Rightarrow \boxed{a \in (-1, 4)}$

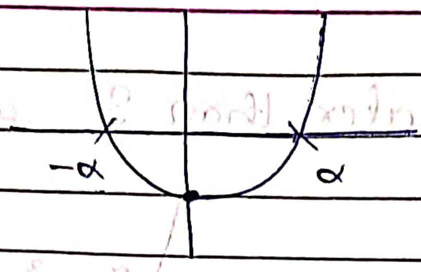
4) $f(0) < 0$



$\boxed{a < (-5)}$

5) i) $f(0) < 0$

ii) $\left(\frac{-B}{2A}\right) = 0$



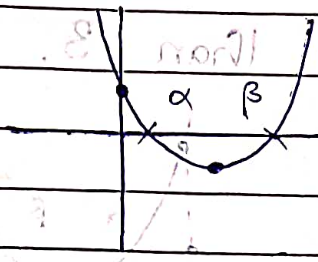
$\left.\begin{array}{l} a<(-5) \\ a=1 \end{array}\right\} \boxed{a \in \phi}$

6) (+ve) roots

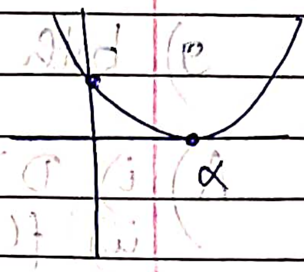
A) i) $D \geq 0$

ii) $\left(\frac{-B}{2A}\right) > 0$

iii) $f(0) > 0$



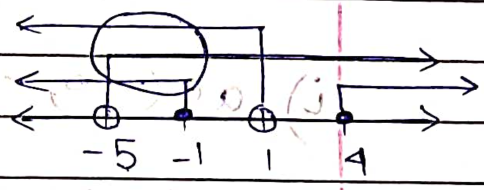
OR



i) $4(a-1)^2 > 4(a+5) \Rightarrow a \in (-\infty, -1] \cup [4, \infty)$

ii) $(a-1) < 0 \Rightarrow a < 1$

iii) $(a+5) > 0 \Rightarrow a > (-5)$



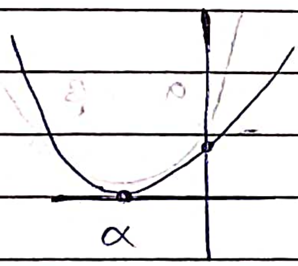
$\Downarrow$

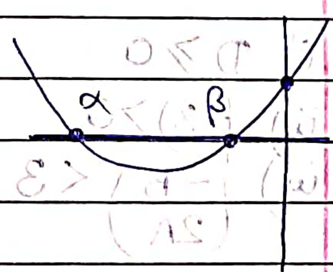
$\boxed{a \in (-5, -1]}$

7) (−ve) roots

A) i) $D \geq 0$

ii) $\left(\frac{-B}{2A}\right) < 0$





iii) $f(0) > 0$

i) $a \in (-\infty, -1] \cup [4, \infty)$, iii) $a > (-5)$

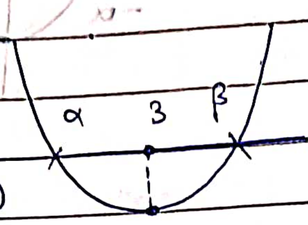
ii) $a > 1 \Rightarrow$

$\Downarrow$

$\boxed{a \in [4, \infty)}$

8) one greater than 3, other smaller than 3.

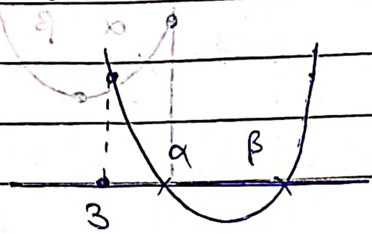
A) $f(3) < 0$

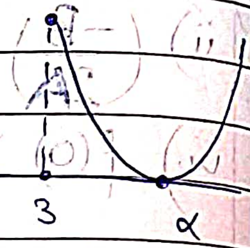


$\Rightarrow 9 + 6(a-1) + (a+5) < 0 \Rightarrow \boxed{a < \left(-8/7\right)}$

9) both greater than 3.

A) i) $D \geq 0$
ii) $f(3) > 0$
iii) $\left(\frac{-B}{2A}\right) > 3$



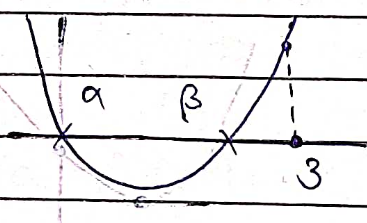


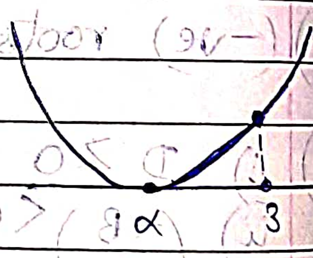
i) $a \in (-\infty, -1] \cup [4, \infty)$ ii) $a > \left(-8/7\right)$

iii) $-(a+1) > 3 \Rightarrow a < (-4) \Rightarrow \boxed{a \in \phi}$

10) both less than 3.

A) i) $D \geq 0$
ii) $f(3) > 0$
iii) $\left(\frac{-B}{2A}\right) < 3$



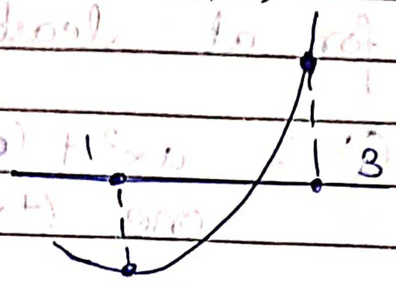


i) $a \in (-\infty, -1] \cup [4, \infty)$ ii) $a > \left(-8/7\right)$ iii) $a > (-4)$

$\Downarrow \Rightarrow \boxed{a \in \left(-8/7, -1\right] \cup [4, \infty)}$

11) exactly one root lies in $(1,3)$:

A) $f(1)f(3) < 0$

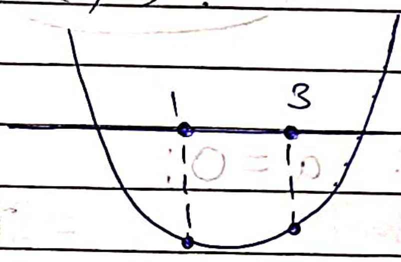


$\Rightarrow (3a+4)(7a+8) < 0$

$\Rightarrow \boxed{a \in (-4/3, -8/7)}$

12) both not lie in $(1,3)$

A) $f(1) < 0$ & $f(3) < 0$



$\Rightarrow a < (-4/3)$ & $a < (-8/7)$

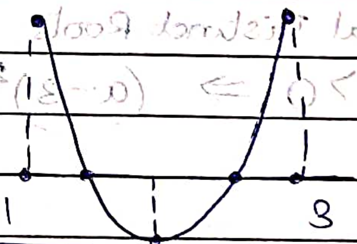
$\Rightarrow \boxed{a < (-4/3)}$

~~13) One > 3 and other < 1~~

13) both root in $(1,3)$

A) • $D \geq 0$

$\Rightarrow a \in (-\infty, -1] \cup [4, \infty)$



• $(-B/2A) \in (1, 3)$

$\boxed{a \in \left(\frac{-8}{7}, -1\right]}$

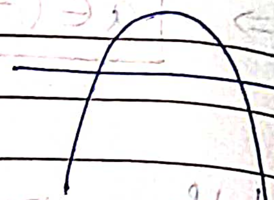
$\Rightarrow -(a-1) \in (1,3) \Rightarrow (a-1) \in (-3,-1) \Rightarrow a \in (-2, 0)$

• $f(1) > 0$ & $f(3) > 0 \Rightarrow a > (-8/7)$

DATE
PAGE

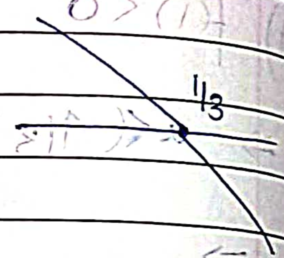
☆Q) find value of 'a' s.t. $ax^2+(a-3)x+1<0$ for at least one (+ve) $x \in R$.

A) $Q \equiv ax^2+(a-3)x+1$ has at least one (+ve) root, if $a \neq 0$.
(both roots distinct)



C1: $a<0$; Always true. (if $x \to \infty$)

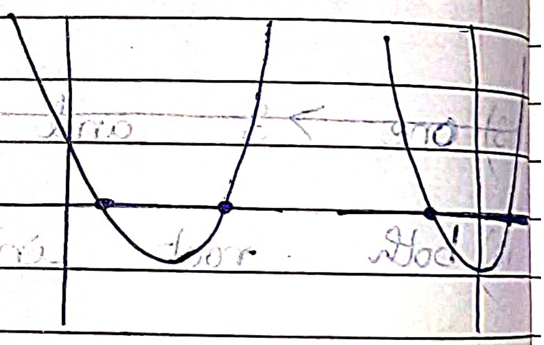
$\Rightarrow$ $a \in (-\infty, 0)$ — (1)



C2: $a=0$; $-3x+1<0$
True $\forall x > 1/3$

$\Rightarrow$ $a=0$ — (2)

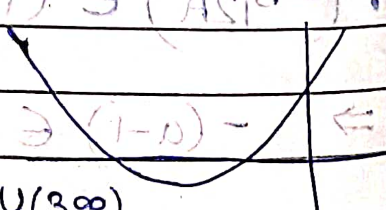
C3: $a>0$;



(At Least 1 (+ve)) = (Real Roots) − (Both −ve)

Real Distinct Roots

- $D>0 \Rightarrow (a-3)^2-4a>0 \Rightarrow a^2-10a+9>0$
$\Rightarrow a \in (-\infty, +1) \cup (9, \infty)$



Both (−ve)

- $D \geq 0 \Rightarrow a \in (-\infty, 1] \cup [9, \infty)$
- $f(0)>0 \Rightarrow 1>0$
- $\left(\frac{-B}{2A}\right)<0 \Rightarrow \left(\frac{a-3}{a}\right)>0 \Rightarrow a \in (-\infty, 0) \cup (3, \infty)$

$\Rightarrow a \in (-\infty, 0) \cup [9, \infty)$

$\Rightarrow$ $a \in (0,1)$ — (3)

Combing (1) $\cup$ (2) $\cup$ (3) $\Rightarrow$ $\boxed{a<1}$

Q) Let $f(x) = x^3 + bx^2 + cx + d$. s.t. $f(0)$ & $f(-1)$ are odd integers. P.T. all roots of $f(x) = 0$ can NOT be integers.

A) $f(0) = d = $ Odd int.

$f(-1) = -1 + b - c + d = $ Odd int. $\Rightarrow$ $\boxed{b - c = \text{Odd int}}$

Let us assume for sake of contradiction that α, β, γ are ~~odd~~ integers satisfying $f(x)$.

$\Rightarrow$ $\alpha + \beta + \gamma = (-b)$, $\sum \alpha\beta = c$, $\prod \alpha = d$

If $\alpha, \beta, \gamma \in \mathbb{Z}$ & $\alpha\beta\gamma = d = $ Odd int.

$\Rightarrow$ $\alpha, \beta, \gamma = $ Odd int.

Now, $\sum \alpha = (-b) = $ odd and $\sum \alpha\beta = c = $ odd int.

$\Rightarrow$ $\boxed{b - c = \text{even.}}$

Hence, contradiction.

$\therefore$ Our assumption is wrong. ■

DATE
PAGE

☆Q) If $b^2 < 2ac$, p.t. $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

A) Let α, β, γ be roots $\Rightarrow \sum\alpha = \left(\frac{-b}{a}\right), \sum\alpha\beta = \left(\frac{c}{a}\right)$

Now, $\left(\frac{b^2 - 2ac}{a^2}\right) = \left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = (\sum\alpha)^2 - 2\sum\alpha\beta$

$\Rightarrow \sum\alpha^2 = b^2 - 2ac < 0$

$\Rightarrow$ At least one complex root
$\Rightarrow$ Exactly 1 real root.

Other Method : $f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = (3a)x^2 + (2b)x + c$

Now, $D(f'(x)) = 4b^2 - 12ac = 4(b^2 - 2ac) < 0$

$\Rightarrow f'(x)$ always (+ve) (or (−ve)) $\Rightarrow f(x)$ always inc (or dec.)

$\Rightarrow$ Exactly 1 real root.

☆Q) Given $a < b < c < d$. P.t. for any $\lambda \in \mathbb{R}$,

$(x-a)(x-c) + \lambda(x-b)(x-d) = 0$ has real roots.

A) Let $f(x) = (x-a)(x-c) + \lambda(x-b)(x-d)$.

C1: $\lambda = 0$, $f(x) = (x-a)(x-c) \Rightarrow$ ✓

C2: $\lambda > 0$, $f(a) = \lambda(b-a)(d-a) > 0$
$f(b) = \boxed{-(b-a)(c-b)} < 0$
$f(d) = \lambda(d-a)(d-c) > 0$

$\Rightarrow$ One root $\in (a, b)$; other $\in (b, d)$

C3: $\lambda < 0$, $f(b) = \boxed{-(b-a)(c-b)} < 0$
$f(c) = (-\lambda)(c-b)(d-c) > 0$
$f(d) = \lambda(d-a)(d-c) < 0$

$\Rightarrow$ One root $\in (b, c)$; other $\in (c, d)$

Q) If $ax^2 + bx + c = 0$ (α) & $-ax^2 + bx + c = 0$ (β), then p.t. $\frac{ax^2}{2} + bx + c$ has a root b/w α & β.

A) Let $f(x) = \frac{ax^2}{2} + bx + c = (ax^2 + bx + c) - (a/2)x^2$
$= (-ax^2 + bx + c) + (3a/2)x^2$

Now, $f(\alpha) = (-a/2)\alpha^2$ & $f(\beta) = (3a/2)\beta^2$.

Observe, $f(\alpha)f(\beta) = \left(-\frac{3a^2}{4}\right)\alpha^2\beta^2 < 0 \Rightarrow$ ✓

196

DATE
PAGE

Q) Find 'a' s.t. $(x^2+x+2)^2-(a-3)(x^2+x+2)(x^2+x+1)+(a-4)(x^2+x+1)^2=0$ has at least one real root.

A) Let $y=\left(\frac{x^2+x+2}{x^2+x+1}\right) \Rightarrow y^2-(a-3)y+(a-4)=0$

$\Rightarrow [y-(a-4)][y-1]=0$

$\Rightarrow y=1, (a-4)$

$\cancel{1}$ as $x\to\infty$ then.

Now range of y, $x^2+x+1 \geq 3/4$

$\Rightarrow \left(\frac{1}{x^2+x+1}\right)\in\left(0,\frac{4}{3}\right]$

$\Rightarrow y=1+\left(\frac{1}{x^2+x+1}\right)\in\left(1,\frac{7}{3}\right]$

$\Rightarrow \quad y=(a-4)\in\left(1,\frac{7}{3}\right] \Rightarrow \boxed{a\in\left(5,\frac{19}{3}\right]}$

Q) Find A & B if $\alpha,\beta,\gamma,\delta$ in H.P.
Given $Ax^2-4x+1=0 \;\begin{matrix}\alpha\\ \gamma\end{matrix}$ & $Bx^2-6x+1=0 \;\begin{matrix}\beta\\ \delta\end{matrix}$

A) Let $\alpha'=1/\alpha$, $\beta'=1/\beta$, $\gamma'=1/\gamma$ & $\delta'=1/\delta$.

$\Rightarrow x^2-4x+A=0 \;\begin{matrix}\alpha'\\ \gamma'\end{matrix}$; $x^2-6x+B=0 \;\begin{matrix}\beta'\\ \delta'\end{matrix}$

and we want $\alpha', \beta', \gamma', \delta'$ in A.P.

Let A.P. ; $p-3q,\ p-q,\ p+q,\ p+3q$

$\Rightarrow \quad \alpha' + \gamma' = 2p - 2q = 4$

$\Rightarrow \quad \beta' + \delta' = 2p + 2q = 6 \quad \Rightarrow \quad p = 5/2, \; q = 1/2$

$\Rightarrow \quad \alpha' = 1, \quad \beta' = 2, \quad \gamma' = 3, \quad \delta' = 4$

$\Rightarrow \quad \boxed{A = 3} \; ; \quad \boxed{B = 8}$

Q) If $(x-3a)(x-a-3) < 0, \; \forall x \in [1, 3]$; find all possible values of 'a'.

A) ~~(a+3) & (3a) lie b/w 1 & 3.~~

~~(a+3) ∈ [~~

1 & 3 lie b/w $(a+3)$ & $3a$.

$\Rightarrow$ Roots of $(x-3a)(x-(a+3))$ outside $[1, 3]$

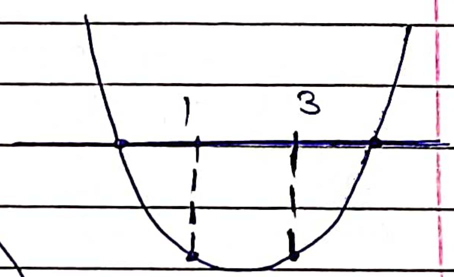
$f(1) < 0 \Rightarrow (3a-1)(a+2) < 0$

$\Rightarrow a \in (-2, 1/3)$

AND

$f(3) < 0 \Rightarrow (a-1)(a) < 0$

$\Rightarrow a \in (0, 1)$



$\boxed{a \in (0, 1/3)}$

Q) Find min. value of $x^2 + 2xy + 3y^2 - 6x - 2y$ for $\forall x, y \in \mathbb{R}$.

A) Let $t = x^2 + x(2y-6) + (3y^2 - 2y)$

$\Rightarrow \quad x^2 + x(2y-6) + (3y^2 - 2y - t) = 0$

198 $x \in \mathbb{R} \Rightarrow (y-3)^2 \geq (3y^2 - 2y - t)$

$\Rightarrow t \geq (2y^2 + 4y - 9)$

DATE

PAGE

$\Rightarrow t \geq 2(y+1)^2 - 7 \Rightarrow \boxed{t_{min} = (-7)}$